Duality Gap Function in Infinite Dimensional Linear Programming*

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Abstract The concept of duality gap function in infinite dimensional linear programming is considered in this paper. Basic properties of the function and two theorems on its behavior are obtained by using duality theorems with interior conditions. As illustrations for the results, we investigate the parametric versions of an example due to D. Gale and parametric linear programs on spaces of continuous functions. The notions of Riemann–Stieltjes integral and function of bounded variation have been shown to be very useful for our investigations.

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1 Introduction

A linear program is an optimization problem in which both the objective and the constraint are described by linear functions. It is finite dimensional if both the number of programming variables and that of constraints are finite, and is infinite dimensional if these two numbers are infinite.

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A rather complete theory for finite dimensional linear programs has been developed. The simplex method for solving such problems, discovered by G.B. Dantzig in 1947, is one of the most famous algorithms of the 20th century. This method has a wide range of applications in optimization. In the last three decades, interior methods have been successfully applied to linear programs to reduce computational time for large-scale problems.

For infinite dimensional programs, a satisfactory theory has not yet been completed in some sense, though there are many interesting results. This is an important area of research, as it has significant applications to continuous transportation, piecewise continuous assignments, time-continuous network-flows, space-continuous flow optimization, optimal design of structures, and so on; see e.g. [1, 6, 12, 15]. This type of problems was first considered by R. Bellman [3] in 1957. His problem was in the context of continuous functions of time and is related to a model of linear optimal control used in production systems. In 1956, R.J. Duffin [5] obtained some foundational results of the theory of infinite dimensional linear programming. Many other authors have further contributed to this theory, including D. Gale, L. Hurwicz, K.O. Kortanek, K.S. Kretschmer, J.M. Borwein, A. Shapiro, C. Zalinescu, etc.

Excellent reviews on duality theorems in infinite dimensional linear programming and some related topics were given by Anderson [1] and by Anderson and Nash [2, pp. 61–63]. In the terminology of Bonnans and Shapiro [4], the problems considered in [2] and in [4, pp. 125–132] are *conic linear optimization problems*. They are more complicated than the *generalized linear programs* [4, pp. 132–145], where the ordering cones are generalized polyhedral convex sets.

Some information about recent achievements in infinite dimensional linear programming in general, and in continuous linear programming in particular, can be found in [18, 19, 20, 21, 22] and the references therein. Note that by introducing a simple right-hand-side perturbation to the constraint systems (to obtain a sequence of dual pairs of relaxed problems), Wu has succeeded in proving a strong duality theorem [22, Theorem 5.1] guaranteeing not only the equality of the optimal values of the primal problem and the dual problem, but also the existence of solutions for these continuous-time linear problems. Back to the past a little bit, a theory about linear semi-infinite programming, where either the number of constraints or the number of variables is finite, was developed by Goberna and López [8]. We also refer to [9] for several related results about linear and nonlinear semi-infinite programming.

Tight connections of duality theory with sensitivity analysis have been discussed by Gretsky et al. [6] and by Shapiro [17]. In particular, in [6], it has been shown that the value function is subdifferentiable at the primal constraint if and only if there exists an optimal dual solution and there is no duality gap.

Further studies of duality for infinite dimensional linear programs and an investigation about the applications of duality theorems in the directions of the last five chapters of [2] and of the paper by Shapiro [17] would be interesting and of importance.

The aim of this paper is twofold. First, to establish basic properties of the duality gap function, which can serve as a tool for qualitative studies of infinite dimensional linear programs. Second, to analyze the example of Gale (see [2, pp. 42-43]) showing that duality gaps do exist for some dual pairs of infinite dimensional linear programs, in parametric forms. In addition, by using the concepts of Riemann–Stieltjes integral and function of bounded variation, we are able to give a series of illustrative examples for our result on the duality gap function of linear programs on standard dual pairs of Banach spaces.

Some preliminaries are given in the next section. The duality gap function is studied in Section 3, where basic properties of the function and two theorems on its behavior are obtained. To illustrate the obtained results, parametric versions of an example due to D. Gale are analyzed in Section 4, and a series of parametric linear programs on spaces of continuous functions are constructed in Section 5.

2 Preliminaries

We begin by recalling the concept of dual pair of topological vector spaces and some related facts (see Anderson and Nash [2], Robertson and Robertson [16], and the references therein).

Definition 2.1 Let X and Y be vector spaces over $\mathbb{R}$. Let be given a bilinear form $\langle \cdot, \cdot \rangle$ such that for each $x \in X$ and for each $y \in Y$ the functionals $\langle x, \cdot \rangle$ and $\langle \cdot, y \rangle$ are linear. If
(i) for each $x \in X \setminus \{0\}$ there exists $y \in Y$ with $\langle x, y \rangle \neq 0$, and
(ii) for each $y \in Y \setminus \{0\}$ there exists $x \in X$ with $\langle x, y \rangle \neq 0$,
then (X, Y) is called a *dual pair*.

One can define a topology on X by letting the sets

$$B_A = \{x \in X \mid -1 \leq \langle x, y \rangle \leq 1, \ \forall\, y \in A\},$$

where A runs through all the finite subsets of Y, to form a base of neighborhoods of the origin. This topology, denoted by $\sigma(X, Y)$, is called the *weak topology* on X. With respect to $\sigma(X, Y)$, X is a locally convex Hausdorff topological vector space. The *weak topology* on Y is defined similarly.

One says that a locally convex topology on X is *consistent* with the dual pair (X, Y) if Y is the dual of X w.r.t. this topology. We know that $\sigma(X, Y)$ is the coarsest one among such topologies on X and there is also the finest one among those called the Mackey topology with notation $\tau(X, Y)$. One constructed $\tau(X, Y)$ as follows. Let $\mathcal{A}$ be the set of all the balanced convex $\sigma(Y, X)$-compact subsets of Y. The sets

$$\varepsilon \left(\bigcap_{1 \leq i \leq k} A_i^0 \right) \qquad (\varepsilon > 0, \ k \in \mathbb{N}),$$

where $A_i^0 := \{x \in X \mid \sup\limits_{y \in A_i} |\langle x, y\rangle| \le 1\}$ with $A_i \in \mathcal{A},\ i = 1, \ldots, k$, form a base of neighborhoods of the origin in the Mackey topology on X. In the special case where X is a normed space, Y is the dual of X, and $\langle x, y\rangle$ is the value of y at x, the Mackey topology on X coincides with its normed topology.

Let (X, Y) and (Z, W) be two dual pairs of vector spaces. In this paper, we always consider the weak topologies $\sigma(X, Y)$ and $\sigma(Z, W)$ unless others are mentioned explicitly. Let $A : X \to Z$ be a $\sigma(X, Y)$–$\sigma(Z, W)$-continuous linear map. The *adjoint* (or *transpose*) of A is the linear map $A^* : W \to Y$ defined by the condition

$$\langle x, A^* w\rangle = \langle Ax, w\rangle \quad \forall x \in X,\ \forall w \in W.$$

It is well known that A^* is $\sigma(W, Z)$–$\sigma(Y, X)$-continuous.

Let P be a convex cone in X. We need $0 \in P$, but we do not require the closedness of P. One says that P is a *positive cone* in the sense that it defines the following partial order in X: Write $x \ge u$ if $x - u \in P$. Let $b \in Z$ and $c \in Y$ be given.

Definition 2.2 The equality constrained linear program corresponding to the data set $\{A,\ P,\ b,\ c\}$ is

$$\text{(EP)} \qquad \min\{\langle x, c\rangle \mid Ax = b,\ x \ge 0\}.$$

A program is called *consistent* if it has at least one feasible solution. The *optimal value* (or simply the *value*) of a consistent program (EP), denoted by val(EP), is defined as the infimum of the objective function $\langle \cdot, c\rangle$ over the set of feasible solutions. If (EP) is inconsistent then, in accordance with the convention $\inf \emptyset = +\infty$, we let $\text{val(EP)} = +\infty$. The optimal solution set (or solution set, for short) of (EP) is denoted by Sol(EP).

Remark 2.1 As in the classical transportation problem (see, e.g., [12, p. 556]), vector c in (EP) can be interpreted as the *cost*, while vector b represents the *supplies* and *demands* in a network optimization problem which is modeled by (EP). Operator A, which represents the *incidence matrix* in the classical transportation problem, can be interpreted as the data describing the *structure* of the network. The constraint $x \ge 0$ (i.e., $x \in P$) can be interpreted as the traditional requirements on the *signs* of the related programming variables. Hence, in our perturbation analysis of (EP), it is reasonable to assume that A and P are fixed, while c and b are subject to change.

The *dual cone* of P is defined by setting $P^* := \{y \in Y \mid \langle x, y\rangle \ge 0,\ \forall x \in P\}$.

Definition 2.3 The *dual problem* of (EP), denoted by (EP*), is

$$\text{(EP}^*\text{)} \qquad \max\{\langle b, w\rangle \mid -A^* w + c \in P^*,\ w \in W\}.$$

If (EP*) is consistent, its optimal value is defined as the supremum of the objective function $\langle b, w\rangle$ over the set of feasible solutions w. One often interprets X as the primal variable space, Z as the primal constraint space, W as the dual variable space, and Y as the dual constraint space.

The next duality theorem can be found in [2, Chap. 3].

Theorem 2.1 (Weak duality) *If* (EP) *and* (EP*) *are both consistent, then* val(EP) $\geq$ val(EP*)*, and both optimal values are finite.*

Note that even if both the primal and dual programs are consistent, one cannot assert that their values are equal (see Section 4 below). This is the main difference between duality theorems in the infinite dimensional setting and duality theorems in the traditional finite dimensional setting.

One says that a program has *no duality gap* if the optimal value of the primal program and that of its dual are equal. There are many conditions guaranteeing the absence of duality gap. We will need two theorems involving the existence of interior points of certain sets.

Theorem 2.2 (Strong duality; see [2, Theorem 3.11]) *Suppose that there is some neighborhood B of the origin in Z with respect to the Mackey topology $\tau(Z, W)$ satisfying $b + B \subset AP$. If* (EP) *has a finite value and there exists $\gamma \in \mathbb{R}$ such that for each $z \in b + B$ one can find $x \in P$ with $Ax = z$ and $\langle x, c\rangle \leq \gamma$, then* (EP) *has no duality gap.*

Since the Mackey topology contains any topology which is consistent with a dual pair, the interior point conditions involving this topology are the weakest ones that can be given in term of consistent topologies. To apply the above theorem, we only need to find an appropriate topology on Z consistent with the dual pair (Z, W), for example the weak topology $\sigma(Z, W)$, such that there is a neighborhood B of the origin with the required properties.

Theorem 2.3 (Strong duality; see [2, Theorem 3.12]) *Let $A : X \to Z$ be a surjective continuous linear map between two Banach spaces X and Z, whose dual spaces are respectively Y and W. If* val(EP) *is finite and there exists $x^0 \in \operatorname{int}(P)$ with $Ax^0 = b$, where* $\operatorname{int}(P)$ *denotes the interior of P, then* (EP) *has no duality gap.*

3 Duality gap function and its basic properties

In this section, we consider the linear program (EP) and its dual (EP*) which are defined in Section 2. However, we will fix A and P, and interpret both b and c as parameters. The reason for doing so has been explained in Remark 2.1.

The function $\varphi : Y \times Z \to \overline{\mathbb{R}} = [-\infty, +\infty]$, with $\varphi(c, b)$ being the infimum of $\langle x, c\rangle$ over the feasible set

$$F(b) := \{x \in X \mid Ax = b,\ x \geq 0\}$$

of (EP), is the *optimal value function* of (EP). The function $\psi : Y \times Z \to \overline{\mathbb{R}}$ with $\psi(c, b)$ being the supremum of $\langle b, w\rangle$ over the feasible set

$$F^*(c) := \{w \in W \mid -A^*w + c \in P^*\}$$

of (EP*), is the *optimal value function* of (EP*).

We now define the concept of duality gap function, which is a function of two variables b and c. Denote by Λ the set of all pairs (c,b) for which both problems (EP) and (EP*) are consistent, i.e.,

$$\Lambda = \{(c,b) \in Y \times Z \mid F^*(c) \neq \emptyset,\ F(b) \neq \emptyset\}.$$

Definition 3.1 The *duality gap function* of (EP) is the function $g : \Lambda \to \mathbb{R}$ with

$$g(c,b) := \varphi(c,b) - \psi(c,b). \tag{1}$$

Remark 3.1 By the weak duality relation in Theorem 2.1, one has

$$-\infty < \psi(c,b) \leq \varphi(c,b) < +\infty, \quad \forall\,(c,b) \in \Lambda.$$

Therefore, $g(c,b)$ is finite and nonnegative for all $(c,b) \in \Lambda$.

Remark 3.2 If $F(b) = \emptyset$ then $\varphi(c,b) = +\infty$. In this case, if $F^*(c) \neq \emptyset$ and if $\psi(c,b) = +\infty$, then the difference $\varphi(c,b) - \psi(c,b) = (+\infty) - (+\infty)$ is undefined. If $F^*(c) = \emptyset$ then $\psi(c,b) = -\infty$; thus $\varphi(c,b) - \psi(c,b) = (+\infty) - (-\infty) = +\infty$. If $F(b) \neq \emptyset$, $\varphi(c,b) = -\infty$, and $F^*(c) = \emptyset$, then the difference $\varphi(c,b) - \psi(c,b) = (-\infty) - (-\infty)$ is undefined.

The above remarks assure us that it is reasonable to study g on the set Λ.

Proposition 3.1 *The effective domain Λ of the duality gap function g is a convex cone which can be computed by the formula*

$$\Lambda = (A^*W + P^*) \times AP. \tag{2}$$

Proof This fact is easy to prove, so we leave it to the reader. Note that (2) yields $(0,0) \in \Lambda$. (The last inclusion can be checked by noting $0 \in F(0)$ and $0 \in F^*(0)$.) $\square$

Proposition 3.2 *The function $g(c,b)$ is nonnegatively homogeneous with respect to each of the variables c and b. In particular, one has $g(0,b) = 0$ and $g(c,0) = 0$ for all $(c,b) \in \Lambda$.*

Proof First, suppose that $c = 0$ and $b \in AP$. On one hand, we have $\varphi(0,b) = 0$ as (EP) is feasible and the objective function is identically zero. On the other hand, by Theorem 2.1, $\psi(0,b) \leq \varphi(0,b)$. Since 0 is a feasible solution of the problem (EP*) corresponding to the pair $(0,b)$, we must have $\psi(0,b) = 0$. Therefore, for $t = 0$, $g(tc,b) = 0 = tg(c,b)$ for all $(c,b) \in \Lambda$. In the same way, we can show that $g(c,tb) = tg(c,b) = 0$ for $t = 0$ and for all $(c,b) \in \Lambda$. Now, let $(c,b) \in \Lambda$ be given arbitrarily. For any $t > 0$, it is clear that $tF^*(c) = F^*(tc)$ and $tF(b) = F(tb)$. Since $F^*(c) \neq \emptyset$ and $F(b) \neq \emptyset$, both sets $F^*(tc)$ and $F(tb)$ are nonempty. Consequently,

$$\varphi(tc,b) = \inf_{x \in F(b)} \langle x, tc \rangle = t \inf_{x \in F(b)} \langle x, c \rangle = t\varphi(c,b)$$

and

$$\psi(tc, b) = \sup_{v \in F^*(tc)} \langle b, v \rangle = t \sup_{w \in F^*(c)} \langle b, w \rangle = t\psi(c, b).$$

It follows that

$$g(tc, b) = \varphi(tc, b) - \psi(tc, b) = tg(c, b)$$

for all $t > 0$. Arguing in the same manner, we obtain

$$g(c, tb) = \varphi(c, tb) - \psi(c, tb) = tg(c, b)$$

for all $t > 0$, which completes the proof. $\square$

A real-valued function defined on a convex subset of a vector space is called a *DC function* if it can be represented as the difference of two convex functions.

Proposition 3.3 *The duality gap function g is a DC function with respect to each variable.*

Proof By Proposition 3.2, $\varphi(c, b)$ and $\psi(c, b)$ are nonnegatively homogeneous w.r.t. each of the variables c, b. Suppose that $b \in AP$ is fixed. For any c_1 and c_2 belonging to $A^*W + P^*$, on one hand one has

$$\begin{aligned}
\varphi(c_1 + c_2, b) &= \inf_{x \in F(b)} \langle c_1 + c_2, x \rangle \\
&\geq \inf_{x \in F(b)} \langle c_1, x \rangle + \inf_{x \in F(b)} \langle c_2, x \rangle = \varphi(c_1, b) + \varphi(c_2, b).
\end{aligned}$$

So $\varphi(\cdot, b)$ is a concave function. On the other hand, as $F^*(c_1) + F^*(c_2) \subset F^*(c_1 + c_2)$, one has

$$\begin{aligned}
\psi(c_1 + c_2, b) &= \sup_{w \in F^*(c_1 + c_2)} \langle b, w \rangle \\
&\geq \sup_{w \in F^*(c_1) + F^*(c_2)} \langle b, w \rangle = \sup_{\substack{w_1 \in F^*(c_1) \\ w_2 \in F^*(c_2)}} \langle b, w_1 + w_2 \rangle \\
&= \sup_{w_1 \in F^*(c_1)} \langle b, w_1 \rangle + \sup_{w_2 \in F^*(c_2)} \langle b, w_2 \rangle = \psi(c_1, b) + \psi(c_2, b).
\end{aligned}$$

Hence $\psi(\cdot, b)$ is a concave function. It follows that

$$g(\cdot, b) = \varphi(\cdot, b) - \psi(\cdot, b) = (-\psi(\cdot, b)) - (-\varphi(\cdot, b))$$

is a DC function.

Now, fix a vector $c \in A^*W + P^*$. For any $b_1, b_2 \in AP$, since $F(b_1) + F(b_2) \subset F(b_1 + b_2)$, one gets

$$\begin{aligned}
\varphi(c, b_1 + b_2) &= \inf_{x \in F(b_1 + b_2)} \langle x, c \rangle \\
&\leq \inf_{x \in F(b_1) + F(b_2)} \langle x, c \rangle = \inf_{\substack{x_1 \in F(b_1) \\ x_2 \in F(b_2)}} \langle x_1 + x_2, c \rangle \\
&= \inf_{x_1 \in F(b_1)} \langle x_1, c \rangle + \inf_{x_2 \in F(b_2)} \langle x_2, c \rangle = \varphi(c, b_1) + \varphi(c, b_2).
\end{aligned}$$

So $\varphi(c,\cdot)$ is a convex function. Furthermore, since

$$\begin{aligned}\psi(c,b_1+b_2) &= \sup_{w\in F^*(c)} \langle b_1+b_2,w\rangle\\ &\le \sup_{w\in F^*(c)} \langle b_1,w\rangle + \sup_{w\in F^*(c)} \langle b_2,w\rangle = \psi(c,b_1)+\psi(c,b_2),\end{aligned}$$

one can assert that $\psi(c,\cdot)$ is a convex function. Hence $g(c,\cdot)=\varphi(c,\cdot)-\psi(c,\cdot)$ is a DC function. □

Definition 3.2 (See e.g. [7, p. 7], [10, Def. 1.8]) Let X be a vector space and M be a nonempty subset of X. The *core* (or the *algebraic interior*) of M is defined by

$$\mathrm{cor}(M) := \{x\in M \mid \forall\, u\in X,\ \exists\,\delta>0 \text{ such that } x+tu\in M,\ \forall\, t\in(-\delta,\delta)\}.$$

Remark 3.3 (See e.g. [7, p. 59], [10, Lemma 1.32]) If X is a finite dimensional normed vector space and $M\subset X$ is a convex set, then $\mathrm{cor}(M)=\mathrm{int}(M)$. If M is a convex subset of a topological vector space with $\mathrm{int}(M)\neq\emptyset$, then we also have $\mathrm{cor}(M)=\mathrm{int}(M)$.

Theorem 2.2 allows us to prove our first theorem on the duality gap function. Here the parameter b is changing while c is fixed.

Theorem 3.1 *Suppose that $c\in A^*W+P^*$ and there exist $b\in Z$, a neighborhood V of the origin in Z with respect to the Mackey topology $\tau(Z,W)$, and a scalar γ such that $b+V\subset AP$, and for each $z\in b+V$ there is $x\in P$ with $Ax=z$ and $\langle x,c\rangle\le\gamma$. Then $g(c,.)$ is identically null on* $\mathrm{cor}(AP)$.

Proof Let c, b, V, γ be given as in the statement of the theorem. Then, by (2) we have $\{c\}\times(b+V)\subset\Lambda$. Hence, for any $\widetilde{b}\in b+V$, both problems

$$\min\{\langle x,c\rangle \mid Ax=\widetilde{b},\ \ x\ge 0\}$$

and

$$\max\{\langle\widetilde{b},w\rangle \mid -A^*w+c\in P^*,\ \ w\in W\}$$

are consistent. For an arbitrarily given vector $b'\in\mathrm{cor}(AP)$, we can find $\mu>0$ such that

$$\widehat{b} := b'+\mu(b'-b) = (1+\mu)b'-\mu b$$

belongs to AP. So there exists $\widehat{x}\in P$ with $A\widehat{x}=\widehat{b}$. By virtue of the convexity of AP and by the fact that $b+V\subset AP$, one has

$$\frac{1}{1+\mu}\widehat{b}+\frac{\mu}{1+\mu}(b+V)\subset AP.$$

Setting $V'=\dfrac{\mu}{1+\mu}V$, we see that V' is a neighborhood of the origin in the topology $\tau(Z,W)$. Since

$$b'+V'=\left(\frac{1}{1+\mu}\widehat{b}+\frac{\mu}{1+\mu}b\right)+\frac{\mu}{1+\mu}V=\frac{1}{1+\mu}\widehat{b}+\frac{\mu}{1+\mu}(b+V),$$

for each $b'' \in b' + V'$ there exists some $\widetilde{b} \in b + V$ such that

$$b'' = \frac{1}{1+\mu}\widehat{b} + \frac{\mu}{1+\mu}\widetilde{b}.$$

As $\widetilde{b} \in b + V$, by assumption there exists $\widetilde{x} \in P$ with $A\widetilde{x} = \widetilde{b}$ and $\langle \widetilde{x}, c\rangle \le \gamma$. Then $A\left(\frac{1}{1+\mu}\widehat{x} + \frac{\mu}{1+\mu}\widetilde{x}\right) = b''$ and

$$\left\langle \frac{1}{1+\mu}\widehat{x} + \frac{\mu}{1+\mu}\widetilde{x}, c \right\rangle = \frac{1}{1+\mu}\langle \widehat{x}, c\rangle + \frac{\mu}{1+\mu}\langle \widetilde{x}, c\rangle \le \gamma',$$

where $\gamma' := \frac{1}{1+\mu}\langle \widehat{x}, c\rangle + \frac{\mu}{1+\mu}\gamma$. Thus, the vector

$$x'' := \frac{1}{1+\mu}\widehat{x} + \frac{\mu}{1+\mu}\widetilde{x}$$

belongs to P, $Ax'' = b''$, and $\langle x'', c\rangle \le \gamma'$. Applying Theorem 2.2 to the pair (c, b'), we have $g(c, b') = 0$. Since $b' \in \mathrm{cor}(AP)$ was taken arbitrarily, it follows that $g(c, .)$ is identically null on $\mathrm{cor}\,(AP)$. $\square$

On the basis of Theorem 2.3, we now establish our second theorem on the duality gap function, where both c and b are subject to change.

Theorem 3.2 *Suppose that X and Z are Banach spaces, Y and W are respectively their dual spaces. Let $A : X \to Z$ be a surjective continuous linear mapping. If there exist $(c, b) \in \Lambda$ and $x^0 \in \mathrm{int}(P)$ such that $Ax^0 = b$, then the duality gap function g is identically null on $(A^*W + P^*) \times \mathrm{cor}(AP)$.*

Proof Given any $(c', b') \in (A^*W + P^*) \times \mathrm{cor}(AP)$, we note that the problem (EP) with (c, b) replaced by (c', b') has a finite optimal value, that is, $|\varphi(c', b')| < \infty$ by Remark 3.1. Since $b' \in \mathrm{cor}(AP)$, there exists $t > 0$ such that $\widehat{b} := b' + t(b' - b) \in AP$. Hence there is $\widehat{x} \in P$ with $A\widehat{x} = \widehat{b}$. As $x^0 \in \mathrm{int}(P)$, there exists a neighborhood V of the origin of X in the normed topology such that $x^0 + V \subset P$. Put

$$x' = \frac{1}{1+t}\widehat{x} + \frac{t}{1+t}x^0.$$

By the convexity of P, $x' \in P$. We have

$$Ax' = \frac{1}{1+t}A\widehat{x} + \frac{t}{1+t}Ax^0 = \frac{1}{1+t}\widehat{b} + \frac{t}{1+t}b = b'.$$

Note that $V' := \frac{t}{1+t}V$ is a neighborhood of the origin in X. Hence, the property

$$x' + V' = \frac{1}{1+t}\widehat{x} + \frac{t}{1+t}(x^0 + V) \subset \frac{1}{1+t}\widehat{x} + \frac{t}{1+t}P \subset P$$

yields $x' \in \mathrm{int}(P)$. We have seen that (EP) with (c, b) being replaced by $(c', b') \in (A^*W + P^*) \times \mathrm{cor}(AP)$ has a finite optimal value and there exists $x' \in \mathrm{int}\, P$ with $Ax' = b'$. The desired property $g(c', b') = 0$ now follows by applying Theorem 2.3. □

Theorems 3.1 and 3.2 create different settings for the study of the duality gap function. The first one allows us to deal with a general situation where X, Y, Z, W are locally convex topological vector spaces. Meanwhile, the second one forces us to restrict our investigations to the Banach space setting, where X and Z are Banach spaces, Y and W are respectively their duals. The restriction is necessary because Theorem 2.3 relies on the Banach open mapping theorem which is not valid for general locally convex Hausdorff topological vector spaces.

The forthcoming two sections will provide us with some illustrative examples for the above theorems.

4 Illustrative examples for Theorem 3.1

The non-parametric example of Gale is available in [2, pp. 42-43]. For a better presentation of the example in the parametric case, we recall here some major facts concerning the original example.

4.1 The example of Gale

Consider the linear program

$$\text{(PG)} \qquad \min\left\{ x_0 \;\middle|\; x_0 + \sum_{i=1}^{\infty} i x_i = 1,\ \sum_{i=1}^{\infty} x_i = 0,\ x_i \geq 0,\ i = 0, 1, 2, \ldots \right\}.$$

The program (PG) is a model of (EP) with the following data. The primal variable space $X = \mathbb{R}^{(\mathbb{N})}$, called *the generalized finite sequence space*, is formed by sequences with finitely many nonzero terms. The dual of X is $Y = \mathbb{R}^{\mathbb{N}}$, which is the space of all real sequences. Note that (X, Y) is a dual pair with respect to the bilinear form

$$\langle x, y \rangle = \sum_{i=0}^{\infty} x_i y_i, \quad \forall\, x = (x_0, x_1, x_2, \ldots) \in X,\ \forall y = (y_0, y_1, y_2, \ldots) \in Y.$$

In what follows, X and Y are considered with the weak topologies (see Section 2). Let $Z = W = \mathbb{R}^2$. Clearly, with respect to the bilinear form

$$\langle z, w \rangle = z_1 w_1 + z_2 w_2, \quad \forall\, z = (z_1, z_2) \in Z,\ \forall w = (w_1, w_2) \in W,$$

(Z, W) is a dual pair. Let $P = \{x = (x_0, x_1, x_2, \ldots) \in X \mid x_i \geq 0,\ i = 0, 1, 2, \ldots\}$, $b = (1, 0) \in Z$, and $c = (1, 0, 0, ...) \in Y$. Let $A : X \to Z$ be the linear map represented by the infinite matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & . & . & . \\ 0 & 1 & 1 & 1 & . & . & . \end{pmatrix} \in M(2 \times \infty).$$

It is known that $\bar{x} := (1, 0, 0, \ldots)$ is the unique feasible solution of (PG), therefore, it is the only optimal solution of (PG). Thus, $\mathrm{Sol(PG)} = \{\bar{x}\}$ and $\mathrm{val(PG)} = 1$. The dual problem of (PG) is

$$(\mathrm{PG}^*) \quad \max \{w_1 \mid w_1 \leq 1,\ iw_1 + w_2 \leq 0,\ i = 1, 2, 3, \ldots,\ w = (w_1, w_2) \in W\}.$$

One has $\mathrm{val}(\mathrm{PG}^*) = 0$ and $\mathrm{Sol}(\mathrm{PG}^*) = \big\{w = (w_1, w_2) \in W \mid w_1 = 0,\ w_2 \leq 0\big\}$. Since $\mathrm{val(PG)} - \mathrm{val}(\mathrm{PG}^*) = 1$, (PG) has a duality gap.

4.2 The example of Gale with both b and c being perturbed

We have seen that for the original example of Gale $b = (1, 0),\ c = (1, 0, 0, ...)$ and the duality gap of (PG) is 1. A natural question arises: *What happens to the duality gap under small perturbations of b and c?* Let us investigate the problem (PG) where both vectors $b = (b_1, b_2) \in Z$ and $c = (c_1, c_2, \ldots) \in Y$ are subject to change. In the notation of the preceding subsection, we consider the parametric program:

$$(\mathrm{PG}_{c,b}) \quad \min \left\{ \sum_{i=0}^{\infty} c_i x_i \;\middle|\; x_0 + \sum_{i=1}^{\infty} i x_i = b_1,\ \sum_{i=1}^{\infty} x_i = b_2,\ x_i \geq 0,\ i = 0, 1, 2, \ldots \right\}.$$

Its dual, denoted by $(\mathrm{PG}^*_{c,b})$, is

$$\max \{b_1 w_1 + b_2 w_2 \mid w_1 \leq c_0,\ iw_1 + w_2 \leq c_i,\ i = 1, 2, 3, ...,\ w = (w_1, w_2) \in W\}.$$

To find the effective domain Λ of the duality gap function g, we will use formula (2). It is not hard to see that

$$AP = \{b = (b_1, b_2) \in Z \mid b_1 \geq b_2 \geq 0\}. \tag{3}$$

The cone $A^*W + P^*$ and its properties can be described as follows.

Claim 1. *Any vector $c \in A^*W + P^*$ is of the form*

$$c = (\alpha_1 + \beta_0, \alpha_1 + \alpha_2 + \beta_1, \ldots, i\alpha_1 + \alpha_2 + \beta_i, \ldots), \tag{4}$$

where $\alpha_j \in \mathbb{R}$ for $j = 1, 2,$ and $\beta_i \geq 0$ for $i \in \mathbb{N}$. It holds that $\mathrm{cor}(A^*W + P^*) = \emptyset$.

The proof of the fact that any $c \in A^*W + P^*$ can be represented as in (4) is left to the reader. Note that the representation (4) for c is not unique. To prove the second part of the claim, suppose on the contrary that $\mathrm{cor}(A^*W + P^*) \neq \emptyset$. Then there exists

$$c = (\alpha_1 + \beta_0, \alpha_1 + \alpha_2 + \beta_1, \ldots, i\alpha_1 + \alpha_2 + \beta_i, \ldots) \in A^*W + P^*,$$

where $\alpha_j \in \mathbb{R}$ for $j = 1, 2,$ and $\beta_i \geq 0$ for $i \in \mathbb{N}$, such that for any $y = (y_0, y_1, \ldots) \in Y \setminus \{0\}$ there exists $\delta = \delta(y) > 0$ with $c + ty \in A^*W + P^*$ for every $t \in (-\delta, \delta)$. Choose $y_i > 0$ for $i \in \mathbb{N}$ such that $\lim\limits_{i\to\infty} \dfrac{i}{y_i} = 0$ and $\lim\limits_{i\to\infty} \dfrac{\beta_i}{y_i} = 0$. Clearly, $y := (y_0, y_1, \ldots)$

belongs to $Y \setminus \{0\}$. Fix one value $t \in (-\delta(y), 0)$. Since $c + ty \in A^*W + P^*$, there exist $\alpha'_j \in \mathbb{R}$, $j = 1, 2$, and $\beta'_i \geq 0$, $i \in \mathbb{N}$, such that

$$i\alpha_1 + \alpha_2 + \beta_i + ty_i = i\alpha'_1 + \alpha'_2 + \beta'_i \tag{5}$$

for $i = 1, 2, \ldots$. Dividing (5) by y_i and taking $\liminf$ of both sides of the resulted equality as $i \to \infty$, we obtain an equality where the left-hand-side is $t < 0$ and the right-hand-side is greater or equal to 0. We have thus arrived at a contradiction, which shows that $\operatorname{cor}(A^*W + P^*) = \emptyset$.

Claim 2. *Suppose that $b = (b_1, 0)$ with $b_1 > 0$ and $c \in A^*W + P^*$ is of the form* (4) *such that*

$$\liminf_{i \to \infty} \left(i^{-1}\beta_i\right) = 0. \tag{6}$$

Then, $g(c, b) > 0$ if $\beta_0 > 0$ and $g(c, b) = 0$ if $\beta_0 = 0$.

Indeed, if $b = (1, 0)$ then the feasible set of $(\mathrm{PG}_{c,b})$ has just one element $x = (1, 0, 0, \ldots)$. Hence, for every vector

$$c = (\alpha_1 + \beta_0, \alpha_1 + \alpha_2 + \beta_1, \ldots, i\alpha_1 + \alpha_2 + \beta_i, \ldots) \in A^*W + P^*,$$

we have $\mathrm{val}(\mathrm{PG}_{c,b}) = \alpha_1 + \beta_0$. If $w = (w_1, w_2)$ is a feasible solution of $(\mathrm{PG}^*_{c,b})$, then

$$-w_1 + \alpha_1 + \beta_0 \geq 0, \quad -iw_1 - w_2 + i\alpha_1 + \alpha_2 + \beta_i \geq 0 \quad (\forall i \geq 1). \tag{7}$$

The dual problem $(\mathrm{PG}^*_{c,b})$ is the following:

$$\max\{w_1 \mid w = (w_1, w_2) \text{ satisfies } (7)\}.$$

By (7),

$$w_1 \leq \alpha_1 + \beta_0 \tag{8}$$

and

$$i(w_1 - \alpha_1) \leq -w_2 + \alpha_2 + \beta_i \quad (i \geq 1). \tag{9}$$

Due to (6), dividing (9) by i and taking $\liminf$ of both sides as $i \to \infty$, we obtain $w_1 \leq \alpha_1$. Choosing $w_1 = \alpha_1$ and $w_2 = \alpha_2$, we see that $w = (\alpha_1, \alpha_2)$ is a feasible solution of $(\mathrm{PG}^*_{c,b})$. Thus $\mathrm{val}(\mathrm{PG}^*_{c,b}) \geq \alpha_1$. Combining this with the inequality $w_1 \leq \alpha_1$, we get $\mathrm{val}(\mathrm{PG}^*_{c,b}) = \alpha_1$. Therefore, for c as in (4) and $b = (0, 1)$ we have

$$g(c, b) = \mathrm{val}(\mathrm{PG}_{c,b}) - \mathrm{val}(\mathrm{PG}^*_{c,b}) = \beta_0.$$

To obtain the desired result for any $b = (b_1, 0)$ with $b_1 > 0$, we use the fact that $g(c, b)$ is homogeneous with respect to b (see Proposition 3.2).

Claim 3. *For every $b = (\beta, \beta)$ with $\beta > 0$, one has $g(c, b) = 0$ for all $c \in A^*W + P^*$.*

To prove this claim, take any $b = (\beta, \beta)$ with $\beta > 0$, and any c satisfying (4). For that b, it is easy to show that $x = (0, \beta, 0, \ldots) \in X$ is the unique feasible solution of $(\mathrm{PG}_{c,b})$. Hence

$$\mathrm{val}(\mathrm{PG}_{c,b}) = \beta(\alpha_1 + \alpha_2 + \beta_1). \tag{10}$$

By the weak duality relation, we have

$$\mathrm{val}(\mathrm{PG}^*_{c,b}) \le \beta(\alpha_1 + \alpha_2 + \beta_1). \tag{11}$$

Since the objective function of $(\mathrm{PG}^*_{c,b})$ is $\beta(w_1 + w_2)$, from (11) it is clear that the proof will be completed if we find a feasible solution $w = (w_1, w_2)$ of the dual problem such that $w_1 + w_2 = \alpha_1 + \alpha_2 + \beta_1$. Assuming the latter, we can rewrite (7) equivalently as

$$w_1 \le \alpha_1 + \beta_0, \quad w_1 \le \alpha_1 + \frac{1}{i-1}(\beta_i - \beta_1) \quad (i \ge 2). \tag{12}$$

Setting

$$w_1 = \min\left\{ \inf\left\{ \alpha_1 + \frac{1}{i-1}(\beta_i - \beta_1) \mid i \ge 2 \right\}, \alpha_1 + \beta_0 \right\}$$

(note that w_1 is finite because $\beta_i \ge 0$ for all $i \in \mathbb{N}$) and $w_2 = (\alpha_1 + \alpha_2 + \beta_1) - w_1$, we see at once that $w := (w_1, w_2)$ satisfies (12). Since $w_1 + w_2 = \alpha_1 + \alpha_2 + \beta_1$ by our choice of w_2, this w is a desired feasible solution for $(\mathrm{PG}^*_{c,b})$.

Claim 4. *For every $c \in A^*W + P^*$ and $b = (b_1, b_2)$ with $b_1 > b_2 > 0$, one has $g(c, b) = 0$.*

To justify this claim, we will rely on Theorem 3.1. For each $b = (b_1, b_2)$ with $b_1 > b_2 > 0$, one can find a unique positive integer n such that $\frac{1}{n+1} \le \frac{b_2}{b_1} < \frac{1}{n}$. Since

$$\frac{1}{n+2} < \frac{b_2}{b_1} < \frac{1}{n},$$

there exists $\varepsilon > 0$ such that the open ball $B(b, \varepsilon)$ with center b and radius ε has the following property

$$B(b, \varepsilon) \subset \left\{ b' = (b'_1, b'_2) \mid b'_1 > 0,\ b'_2 > 0,\ \frac{1}{n+2} < \frac{b'_2}{b'_1} < \frac{1}{n} \right\}.$$

For each vector $b' = (b'_1, b'_2)$ from the ball $B(b, \varepsilon)$ we define a vector

$$x = (x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots) \in X$$

as follows: let $x_i = 0$ for every $i \notin \{1, n+2\}$, and (x_1, x_{n+2}) be the unique solution of the linear system

$$\begin{cases} x_1 + (n+2)x_{n+2} & = b'_1 \\ x_1 + x_{n+2} & = b'_2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_{n+2} & = \dfrac{b'_1 - b'_2}{n+1}, \\ x_1 & = \dfrac{(n+2)b'_2 - b'_1}{n+1}. \end{cases}$$

Since $x_{n+2} > 0$ and $x_1 > 0$, $x = (x_0, x_1, x_2, \ldots, x_{n+1}, x_{n+2}, \ldots)$ defined as above is a feasible solution of $(\mathrm{PG}_{c,b})$. For any c of the form (4), one has

$$\langle x, c \rangle = (\alpha_1 + \alpha_2 + \beta_1)\frac{(n+2)b'_2 - b'_1}{n+1} + [(n+2)\alpha_1 + \alpha_2 + \beta_{n+2}]\frac{b'_1 - b'_2}{n+1}.$$

As the expression on the right-hand-side of this equality is bounded above by a real constant γ when $b' = (b'_1, b'_2)$ moves within the open ball $B(b, \varepsilon)$, by Theorem 3.1 we can assert that $g(c, b') = 0$ for each $b' \in \mathrm{cor}(AP)$. In particular, we have $g(c, b) = 0$.

Claim 5. *For $b = (0, 0)$, one has $g(c, b) = 0$ for all $c \in A^*W + P^*$.*

This claim can be proved easily.

Summing up the above five claims, we get formulas for computing the duality gap function $g(c, b)$ for all $(c, b) \in \Lambda$, provided that (6) is satisfied when $b = (b_1, 0)$, $b_1 > 0$.

In connection with Claim 2, we observe that the question whether the result is valid without the extra assumption (6) remains open.

Remark that the example of Gale with parameters can be also analyzed by the model of countable semi-infinite program in [2, Section 4.2] and Theorem 4.2 from [2, p. 67].

We intend to consider applications of Theorem 3.1 to the duality gap functions related to the well-known example of Kretschmer (see [13, p. 230], [2, pp. 43-45]) and another interesting problem from [2, p. 52], which is denoted by (P0), in a subsequent paper.

5 Illustrative examples for Theorem 3.2

This section provides some examples of (EP) with $\mathrm{int}P \neq \emptyset$ where Theorem 3.2 can be applied to study the duality gap and the duality gap function.

Let α and β be two real numbers, $\alpha < \beta$; $X = C[\alpha, \beta]$, the space of all real-valued continuous functions on $[\alpha, \beta]$. By the Riesz representation theorem, the dual space of X is $Y = NBV[\alpha, \beta]$, the space of all *functions of bounded variation* on $[\alpha, \beta]$ which vanish at α and which are continuous from the left at every point in (α, β); see e.g. [11] or [14]. It is known that X is a Banach space equipped with the supremum (or Chebyshev) norm

$$||x|| = \max_{\alpha \le t \le \beta} |x(t)|$$

and Y is a Banach space with the norm

$$||y|| = \mathrm{T.V.}(y)$$

where $\mathrm{T.V.}(y)$ denotes the *total variation* of y.

Consider the problem

$$\text{(P1)} \qquad \min \left\{ \int_\alpha^\beta x(t)\, dc(t) \;\middle|\; \int_\alpha^\beta x(t)\, da_i(t) = b_i,\ i = \overline{1, m};\ x(t) \ge 0 \quad \forall t \in [\alpha, \beta] \right\}.$$

Here we work with *Riemann–Stieltjes integrals*. Both the primal constraint space Z and its dual space W can be identified with $\mathbb{R}^m$. The functions c, a_i for all $i = \overline{1, m}$

are fixed elements of Y. Vector $b = (b_1, b_2, \dots, b_m)$ is taken from Z. The positive cone $P \subset X$ is

$$P = \{x \in X \,|\, x(t) \geq 0 \ \ \forall t \in [0,1]\}.$$

It is not hard to see that P has nonempty interior, moreover,

$$\operatorname{int} P = \{x \in X \mid x(t) > 0 \ \ \forall t \in [\alpha, \beta]\}\,. \tag{13}$$

Let $A : X \to Z$ be defined by

$$Ax = \left(\int_\alpha^\beta x(t)da_1(t), \int_\alpha^\beta x(t)da_2(t), \dots, \int_\alpha^\beta x(t)da_m(t) \right) \quad \forall x \in X.$$

Notice that A is a continuous linear operator from X to Z. In addition, one can easily prove that A is surjective if and only if the set $\{a_1, a_2, \dots, a_m\}$ is linearly independent in $NBV[\alpha, \beta]$.

Claim 1. *We have*

$$A^*W + P^* = \left\{ \sum_{i=1}^m w_i a_i + y \;\middle|\; w = (w_1, w_2, \dots, w_m) \in W, \ y \in M \right\}, \tag{14}$$

where M is the set of all nondecreasing functions in Y.

Indeed, the fulfillment of the equality $\langle x, A^*w \rangle = \langle Ax, w \rangle$ for all $x \in X$ and $w \in W$ implies that

$$A^*w = \sum_{i=1}^m w_i a_i, \quad \forall w = (w_1, w_2, \dots, w_m) \in W.$$

So, to obtain (14), it suffices to show that $P^* = M$. Since

$$P^* = \left\{ y \in Y \mid \int_\alpha^\beta x(t)dy(t) \geq 0 \quad \forall x \in P \right\},$$

from the definition of Riemann–Stieltjes integral it follows that $M \subseteq P^*$. It remains to prove that the strict inclusion cannot occur. If there exists a $y \in P^* \setminus M$, then we would find distinct points $t_1, t_2 \in [\alpha, \beta]$, $t_1 < t_2$, such that $\mu := y(t_1) - y(t_2)$ is positive. We will construct a function $x \in P$ satisfying $\int_\alpha^\beta x(t)dy(t) < 0$. Let $\delta \in (0, t_2 - t_1)$ be given arbitrarily. Put

$$x(t) = \begin{cases} 0 & \text{for } t \in [\alpha, t_1 - \delta] \cup (t_2, \beta] \\ \delta^{-1}t + 1 - \delta^{-1}t_1 & \text{for } t \in (t_1 - \delta, t_1] \\ 1 & \text{for } t \in (t_1, t_2 - \delta] \\ -\delta^{-1}t + \delta^{-1}t_2 & \text{for } t \in (t_2 - \delta, t_2]. \end{cases}$$

It is clear that $x \in P$. Setting $A_\delta = \int\limits_{t_1-\delta}^{t_1} x(t)dy(t)$, $B_\delta = \int\limits_{t_1}^{t_2-\delta} dy(t)$, and $C_\delta = \int\limits_{t_2-\delta}^{t_2} x(t)dy(t)$, we have

$$\int\limits_{\alpha}^{\beta} x(t)dy(t) = A_\delta + B_\delta + C_\delta. \tag{15}$$

Note that

$$B_\delta = \int\limits_{t_1}^{t_2-\delta} dy(t) = y(t_2-\delta) - y(t_1) \xrightarrow{\delta\searrow 0} -\mu = y(t_2) - y(t_1) < 0.$$

In addition, one has

$$|A_\delta| \leq \max_{t_1-\delta\leq t\leq t_1} |x(t)|\,\text{T.V.}\left(y_{|_{[t_1-\delta,t_1]}}\right) = \text{T.V.}\left(y_{|_{[t_1-\delta,t_1]}}\right) \xrightarrow{\delta\searrow 0} 0,$$

where $y_{|_{[t_1-\delta,t_1]}}$ stands for the restriction of y on $[t_1-\delta, t_1]$. Similarly,

$$|C_\delta| \leq \max_{t_2-\delta\leq t\leq t_2} |x(t)|\,\text{T.V.}\left(y_{|_{[t_2-\delta,t_2]}}\right) = \text{T.V.}\left(y_{|_{[t_2-\delta,t_2]}}\right) \xrightarrow{\delta\searrow 0} 0.$$

Therefore, we can find $\delta \in (0, t_2 - t_1)$ such that $B_\delta \leq -\frac{\mu}{2}$, $|A_\delta| \leq \frac{\mu}{6}$, $|C_\delta| \leq \frac{\mu}{6}$. With that δ, by (15) we obtain

$$\int\limits_{\alpha}^{\beta} x(t)dy(t) \leq -\frac{\mu}{6} < 0,$$

which shows that $y \notin P^*$, a contradiction.

Claim 2. *The duality gap g is identically zero on $(A^*W + P^*) \times \operatorname{int}(AP)$, provided that $\{a_1, a_2, \ldots, a_m\}$ is linearly independent. In particular, $g(c', b') = 0$ for every pair (c', b') with $c' \in \left\{\sum\limits_{i=1}^{m} w_i a_i + y \;\middle|\; w = (w_1, w_2, \ldots, w_m) \in W,\ y \in M\right\}$, where M is the set of all nondecreasing functions in Y, and $b' \in A(\operatorname{int}P)$.*

Indeed, since $\{a_1, a_2, \ldots, a_m\}$ is linearly independent, A is surjective. For any positive continuous function x_0 defined on $[\alpha, \beta]$, one has $x_0 \in \operatorname{int}(P)$ (since (13)). Therefore, with $b := Ax_0$ and $c \in A^*W + P^*$, we see that $(c, b) \in \Lambda = (A^*W + P^*) \times AP$ and all the assumptions of Theorem 3.2 are satisfied. Since AP is a convex set in $\mathbb{R}^m$, $\operatorname{cor}(AP) = \operatorname{int}(AP)$. Hence, the first assertion of our claim is valid. To prove the second assertion, we observe that $A(\operatorname{int}P)$ is an open set by the Banach open mapping theorem. So $A(\operatorname{int}P) \subset \operatorname{int}(AP)$. Now it is clear that the second assertion follows from the first one and the surjectivity of A.

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